MAXIMIZING VISIBILITY IN NONCONVEX POLYGONS: NONSMOOTH ANALYSIS AND GRADIENT ALGORITHM DESIGN*

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Abstract. This paper presents a motion control algorithm for a planar mobile observer such as, e.g., a mobile robot equipped with an omni-directional camera. We propose a nonsmooth gradient algorithm for the problem of maximizing the area of the region visible to the observer in a simple nonconvex polygon. First, we show that the visible area is almost everywhere a locally Lipschitz function of the observer location. Second, we provide a novel version of LaSalle Invariance Principle for discontinuous vector fields and Lyapunov functions with a finite number of discontinuities. Finally, we establish the asymptotic convergence properties of the nonsmooth gradient algorithm and we illustrate numerically its performance.

1. Introduction. Consider a single-point mobile robot in a planar nonconvex environment modeled as a simple polygon: how should the robot move in order to monotonically increase the area of its visible region (i.e., the region within its line of sight)? This problem is the subject of this paper, together with the following modeling assumptions. The dynamical model for the robot's motion is a first order system of the form $\dot{p} = u$, where p refers to the position of the robot in the environment and u is the driving input. The robot is equipped with an omni-directional camera and range sensor; the range of the sensor is larger than the diameter of the environment. The robot does not know the entire environment and its position in it, and its instantaneous motion depends only on what is within line of sight (this assumption restricts our attention to memoryless feedback laws).

In broad terms, this problem is related to numerous optimal sensor location and motion planning problems in the computational geometry, geometric optimization, and robotics literature. In computational geometry [6], the classical Art Gallery Problem amounts to finding the optimum number of guards in a nonconvex environment so that each point of the environment is visible by at least one guard. A heuristic for this problem is to use a greedy approach wherein the first robot (guard) is placed at the point where it sees the maximum area. The next robot is placed where it sees the maximum area not visible to the first and so on. In robotics, this approach is useful for 2D map building wherein a robot moves in such a way so that its next position is the best in terms of what it can see additionally. In this robotic context, these problems are referred to as Next Best View problems. The specific problem of interest in this paper is that of optimally locating a guard in a simple polygon. To the best of our knowledge, this problem is still open and is the subject of ongoing research; see [11, 16, 3], and the surveys on geometric optimization and art gallery problems [1, 14]. However, randomized algorithms for finding the optimal location up to a constant factor approximation exist; see [3]. These algorithms can be regarded as

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open-loop algorithms that require knowledge of the environment. Closed-loop heuristic algorithms for the Next Best View problem are proposed and simulated in [9] and in the early work [12]. This problem also derives its motivation from the behavior of certain territorial animals. A particularly relevant reference is the study of effect of visibility on space use by red-capped cardinals [7]. These are birds that defend territories along shorelines of rivers and lakes and tend to spend the majority of their time near peninsulas (areas that offer greater amount of visibility of their respective territories) rather than bays.

A second set of relevant references are those on nonsmooth stability analysis. Indeed, our approach to maximizing visible area is to design a nonsmooth gradient flow. To define our proposed algorithm we rely on the notions of generalized gradient [4] and of Filippov solutions for differential inclusions [8]. To study our proposed algorithm we extend recent results on the stability and convergence properties of nonsmooth dynamical systems, as presented in [15, 2].

The contributions of this paper are threefold. First, we prove some basic properties of the area visible from a point observer in a nonconvex polygon Q, see Figure 1.1.

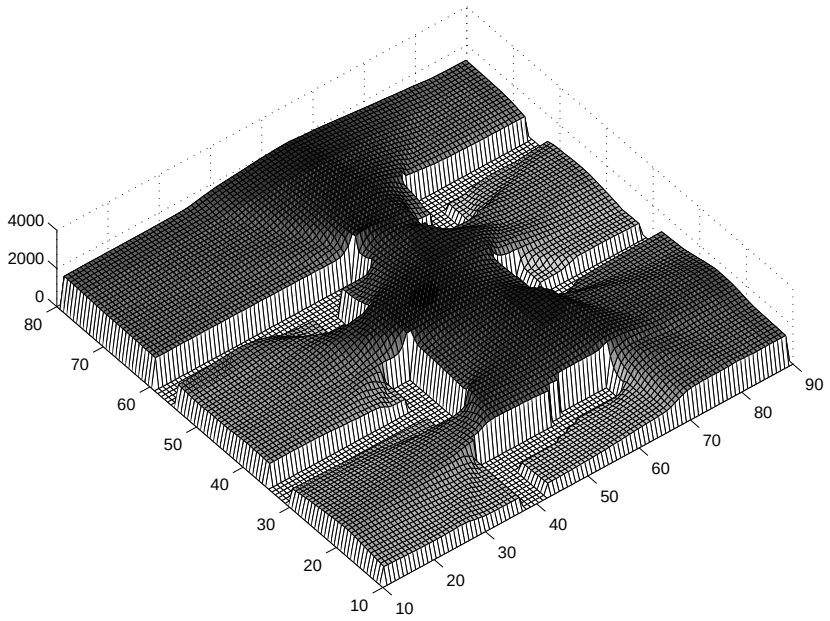

FIG. 1.1. *The visible area function over a nonconvex polygon.*

Namely, we show that the area of the visibility polygon, as a function of the observer position, is a locally Lipschitz function almost everywhere, and that the finite point set of discontinuities consists of the reflex vertices of the polygon Q. Additionally, we compute the generalized gradient of the function and show that it is, in general not regular. Second, we provide a generalized version of the certain stability theorems for discontinuous vector fields available in the literature [15, 2]. Specifically, we provide a generalized nonsmooth LaSalle Invariance Principle for discontinuous vector fields, Filippov solutions, and Lyapunov functions that are locally Lipschitz almost everywhere (except for a finite set of discontinuities). Third and last, we use these novel results to design a nonsmooth gradient algorithm that monotonically

increases the area visible to a point observer. To the best of our knowledge, this is the first provably correct algorithm for this version of the Next Best View problem. We illustrate the performance of our algorithm via simulations for some interesting polygons.

The paper is organized as follows. Section 2 contains the analysis of the smoothness and of the generalized gradient of the function of interest. Section 3 contains the novel results on nonsmooth stability analysis. Section 4 presents the nonsmooth gradient algorithm and the properties of the resulting closed-loop system. Finally, the simulations in Section 5 illustrate the convergence properties of the algorithm.

2. The area visible from an observer. In this section we study the area of the region visible to a point observer equipped with an omnidirectional camera. We show that the visible area, as a function of the location of the observer, is locally Lipschitz, except at a finite point set. We prove that, for general nonconvex polygons, the function is not regular. We also provide expressions for the generalized gradient of the visible area function wherever it is locally Lipschitz. We refer the reader to Appendix A for the notion of locally Lipschitz functions and related concepts.

Let us start by introducing the set of lines on the plane $\mathbb{R}^2$. For $(a,b,c) \in \mathbb{R}^3 \setminus \left\{(0,0,c) \in \mathbb{R}^3 \mid c \in \mathbb{R}\right\}$, define the equivalence class $[(a,b,c)]$ by

$$[(a,b,c)] = \left\{(a',b',c') \in \mathbb{R}^3 \mid (a,b,c) = \lambda(a',b',c'),\ \lambda \in \mathbb{R}\right\}.$$

The *set of lines on* $\mathbb{R}^2$ is defined as

$$\mathbb{L} = \left\{[(a,b,c)] \subset \mathbb{R}^3 \mid (a,b,c) \in \mathbb{R}^3,\ a^2+b^2 \neq 0\right\}.$$

It is possible to show that $\mathbb{L}$ is a 2-dimensional manifold, sometimes referred to as the affine Grassmannian of lines in $\mathbb{R}^2$; see [10].

Next, two simple and useful functions are introduced. Let $f_{\text{pl}} : \mathbb{R}^2 \times \mathbb{R}^2 \setminus \left\{(p,p) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid p \in \mathbb{R}^2\right\} \to \mathbb{L}$ map two distinct points in $\mathbb{R}^2$ to the line passing through them. For $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, the function f_{pl} admits the expression

$$f_{\text{pl}}\left((x_1,y_1),(x_2,y_2)\right) = [(y_2-y_1, x_1-x_2, y_1x_2 - x_1y_2)].$$

If $l_1 \parallel l_2$ denotes that the two lines $l_1, l_2 \in \mathbb{L}$ are parallel, let $f_{\text{lp}} : \mathbb{L}^2 \setminus \{(l_1,l_2) \in \mathbb{L}^2 \mid l_1 \parallel l_2\} \to \mathbb{R}^2$ map two lines that are not parallel to their unique intersection point. Given two lines $[(a_1,b_1,c_1)]$ and $[(a_2,b_2,c_2)]$ that are not parallel, the function f_{lp} admits the expression

$$f_{\text{lp}}\left([(a_1,b_1,c_1)],[(a_2,b_2,c_2)]\right) = \left(\frac{b_2c_1 - b_1c_2}{a_2b_1 - a_1b_2}, \frac{a_1c_2 - a_2c_1}{a_2b_1 - a_1b_2}\right).$$

Note that the functions f_{pl} and f_{lp} are class C^ω, i.e., they are analytic over their domains.

Now, let us turn our attention to the polygonal environment. Let Q be a simple polygon, possibly nonconvex. A polygon is said to be simple if the only points in the plane belonging to two polygon edges are the polygon vertices. Such a polygon has a well defined interior and exterior. *Note that a simple polygon can contain holes.* Let $\mathring{Q}$ and ∂Q denote the interior and the boundary of Q, respectively. Let $\text{Ve}(Q) = (v_1,\ldots,v_n)$ be the list of vertices of Q ordered counterclockwise. The *interior angle of a vertex* v of Q is the angle formed inside Q by the two edges of the boundary of Q incident at v. The point $v \in \text{Ve}(Q)$ is a *reflex vertex* if its interior angle is strictly

greater than π. Let $\mathrm{Ve}_r(Q)$ be the list of reflex vertices of Q. If S is a finite set, then let $|S|$ denote its cardinality.

A point $q \in Q$ is *visible from* $p \in Q$ if the segment between q and p is contained in Q. The *visibility polygon* $S(p) \subset Q$ from a point $p \in Q$ is the set of points in Q visible from p. It is convenient to think of $p \mapsto S(p)$ as a map from Q to the set of polygons contained in Q. It must be noted that the visibility polygon is not necessarily a simple polygon.

DEFINITION 2.1. *Let v be a reflex vertex of Q, and let $w \in \mathrm{Ve}(Q)$ be visible from v. The* (v, w)-generalized inflection segment *$I(v, w)$ is the set*

$$I(v, w) = \{q \in S(v) \mid q = \lambda v + (1 - \lambda)w, \lambda \geq 1\}.$$

A reflex vertex v of Q is an anchor of $p \in Q$ *if it is visible from p and if $\{q \in S(v) \mid q = \lambda v + (1 - \lambda)p, \lambda > 1\}$ is not empty.*

In other words, a reflex vertex is an anchor of p if it occludes a portion of the environment from p. Figure 2.1 illustrates the various quantities defined above. Given a point q and a line l, let $\mathrm{dist}(q, l)$ denote the distance between them.

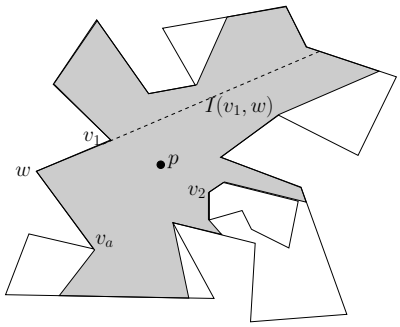

FIG. 2.1. *Reflex vertices v_1 and v_2, a generalized inflection segment $I(v_1, w)$, an anchor v_a of p and the visibility polygon (shaded region) from p. Note that the polygonal environment has a hole.*

THEOREM 2.2. *Let $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ be the set of generalized inflection segments of Q, and let P be a connected component of $Q \backslash \bigcup_{\alpha \in \mathcal{A}} I_\alpha$ For all $p \in P$, the visibility polygon $S(p)$ is simple and has a constant number of vertices, say $\mathrm{Ve}(S(p)) = \{u_1(p), \ldots, u_k(p)\}$. For all $i \in \{1, \ldots, k\}$, the map $P \ni p \mapsto u_i(p)$ is C^ω and*

$$\mathrm{d}u_i(p) = \begin{cases} 0, & u_i(p) \in \mathrm{Ve}(Q), \\ \dfrac{\mathrm{dist}(v_a, l)}{(\mathrm{dist}(p, l) - \mathrm{dist}(v_a, l))^2 \sqrt{a^2 + b^2}} \begin{bmatrix} -b \\ a \end{bmatrix} \begin{bmatrix} y - y_a \\ x_a - x \end{bmatrix}^T, & u_i(p) = f_{\mathrm{lp}}(f_{\mathrm{pl}}(v_a, p), l), \end{cases}$$

where $v_a = (x_a, y_a)$ is an anchor of p and where $l = [(a, b, c)]$ is a line defined by an edge of Q.

Proof. The first part of the proof is by contradiction. Let $|\mathrm{Ve}(S(p'))| > |\mathrm{Ve}(S(p))|$ for some point $p' \in P$. This means that at least one additional vertex is visible from p' that was occluded by an anchor of p. Two cases may arise. First, when the additional vertex belongs to $\mathrm{Ve}(Q)$, then by our definition, p and p' must lie on opposite sides of a generalized inflection segment. This is a contradiction. Secondly, if the additional vertex does not belong to $\mathrm{Ve}(Q)$, it must be the projection of a reflex vertex (acting as an anchor). Here again two cases may arise: (1) the reflex vertex is visible from p, and (2) it is not. The first case is possible only if the reflex vertex is visible but

does not act as an anchor. So, positive lengths of both sides adjoining the reflex vertex must also be visible from p and at least one of the sides is completely not visible from p' since there is a projection. This means that p and p' lie on opposite sides of a generalized inflection segment generated by the reflex vertex and one of its adjacent vertices. This is a contradiction. The second case is possible if the reflex vertex in question is occluded by another reflex vertex. But this means that p and p' lie on opposite sides of the generalized inflection segment from the reflex vertex to the anchor occluding the reflex vertex; again this is a contradiction. If, on the other hand, $|\operatorname{Ve}(S(p'))| < |\operatorname{Ve}(S(p))|$, then the above arguments hold by interchanging p and p'. Hence, p and p' lie on opposite sides of a generalized inflection segment which is a contradiction. This completes the proof that $|\operatorname{Ve}(S(p'))|$ is constant for all $p' \in P$.

Let $p \in P$. Since the visibility polygon $S(p)$ is star-shaped and since any ray emanating from p can intersect $S(p)$ at most at two distinct points, then $S(p)$ is simple. (Indeed, if the ray emanating from p intersects the environment at three or more points inside $S(p)$, then p must belong to a generalized inflection segment.)

Regarding the second statement, it is clear that if $u_i(p)$ is a vertex of Q then it is independent of p. Instead, if $u_i(p) \notin \operatorname{Ve}(Q)$, then

$$u_i(p) = f_{\mathrm{lp}}(f_{\mathrm{pl}}((x,y),(x_a,y_a)),\ell)$$

where $p = (x,y)$, $v_a = (x_a,y_a)$ is an anchor of p, and ℓ is the line, determined by an edge of Q, that identifies u_i. Now, $p \in P$ implies $p \neq v_a$. It follows that $f_{\mathrm{pl}}(p,v_a)$ is C^ω for all $p \in P$. Also, from the definition of $u_i(p)$, it is clear that $f_{\mathrm{pl}}(p,v_a) \nparallel \ell$. Therefore, for all $p \in P$, $f_{\mathrm{lp}}(f_{\mathrm{pl}}(p,v_a),\ell)$ is C^ω; this implies that $p \mapsto u_i(p)$ is also C^ω. The formula for the derivative can be verified directly. □

Next, the area of a visibility polygon as a function of the observer location is studied, see Figure 1.1. Recall that the area of a simple polygon Q with counterclockwise-ordered vertices $\operatorname{Ve}(Q) = ((x_1,y_1),\ldots,(x_n,y_n))$ is given by

$$A(Q) = \frac{1}{2}\sum_{i=1}^{n} x_i(y_{i-1} - y_{i+1}),$$

where $(x_0,y_0) = (x_n,y_n)$ and $(x_{n+1},y_{n+1}) = (x_1,y_1)$. As in the previous theorem, let $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ be the set of generalized inflection segments of Q and let P be a connected component of $Q \setminus \bigcup_{\alpha\in\mathcal{A}} I_\alpha$. Next, if $p \in P$, the visibility polygon from p has a constant number of vertices, say $k = |\operatorname{Ve}(S(p))|$, is simple, and satisfies $A\circ S(p) = \sum_{i=1}^{k} x_i(y_{i-1} - y_{i+1})$ where $\operatorname{Ve}(S(p)) = (u_1,\ldots,u_k)$ are ordered counterclockwise, $u_i(p) = (x_i,y_i)$, $u_0 = u_k$, and $u_{k+1} = u_1$. Therefore, $P \ni p \mapsto A\circ S(p)$ is also C^ω and

$$\mathrm{d}(A\circ S)(p) = \sum_{i=1}^{k} \frac{\partial A(u_1,\ldots,u_k)}{\partial u_i}\mathrm{d}u_i(p). \tag{2.1}$$

Remark 2.3. For any $u_i(p) \notin \operatorname{Ve}(Q)$, we have

$$\frac{\partial(A\circ S)}{\partial u_i}\mathrm{d}u_i(p) = \frac{\operatorname{dist}(v_a,l)}{2}\,\frac{\operatorname{dist}(u_{i+1},l) - \operatorname{dist}(u_{i-1},l)}{(\operatorname{dist}(p,l) - \operatorname{dist}(v_a,l))^2}\begin{bmatrix} y - y_a \\ x_a - x \end{bmatrix}^T. \tag{2.2}$$

Note here that $\frac{\partial(A\circ S)}{\partial u_i}\mathrm{d}u_i(p)$ is perpendicular to $p - v_a$.

To illustrate (2.1) and (2.2), it is convenient to introduce the *versor* operator defined by $\operatorname{vers}(X) = X/\|X\|$ if $X \in \mathbb{R}^2 \setminus \{0\}$ and by $\operatorname{vers}(0) = 0$. We depict the normalized gradient $\operatorname{vers}(\mathrm{d}(A\circ S))$ of the visible area function in Figure 2.2.

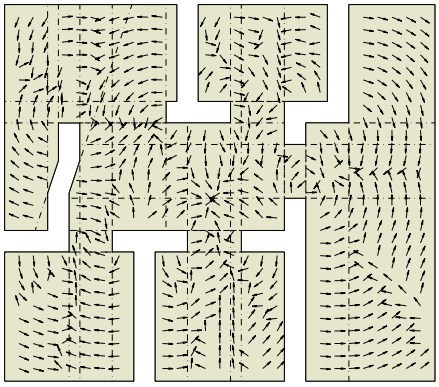

FIG. 2.2. *Normalized gradient of the visible area function over the nonconvex polygon depicted in Figure 1.1. The dashed lines represent some of the generalized inflection segments.*

THEOREM 2.4. *The map $A \circ S$ restricted to $Q \setminus \mathrm{Ve}_r(Q)$ is locally Lipschitz.*

Proof. By Theorem 2.2, it suffices to consider points lying on generalized inflection segments. Let p belongs to multiple, say m, generalized inflection segments $\{I_\alpha\}_{\alpha\in\{1,\dots,m\}}$. Let $B(p,\epsilon)$ be the open ball of radius ϵ centered at p; let ϵ be small enough such that no generalized inflection segments intersect $B(p,\epsilon)$ other than $\{I_\alpha\}_{\alpha\in\{1,\dots,m\}}$. For $\alpha \in \{1,\dots,m\}$, let v_{k_α} be the anchor determining the generalized inflection segment I_α. Without loss of generality, it can be assumed that no anchor is visible from p other than $v_{k_1},\dots,v_{k_m}$. For $\alpha \in \{1,\dots,m\}$, lines $l_\alpha \perp f_{\mathrm{pl}}(p, v_{k_\alpha})$ can be constructed with the property that $l_\alpha \cap Q = \emptyset$ and the vector $v_{k_\alpha} - p$ points toward l_α. Let, h_α be the line parallel to l_α, tangent to $B(\epsilon, p)$, and intersecting the segment from p to v_{k_α}. Let p' and p'' belong to $B(p,\epsilon) \cap (Q \setminus \mathrm{Ve}_r(Q))$. Next, let $q'_\alpha = f_{\mathrm{lp}}(f_{\mathrm{pl}}(p', v_{k_\alpha}), l_\alpha)$ and $q''_\alpha = f_{\mathrm{lp}}(f_{\mathrm{pl}}(p'', v_{k_\alpha}), l_\alpha)$; see Figure 2.3. Let v'_α and v''_α

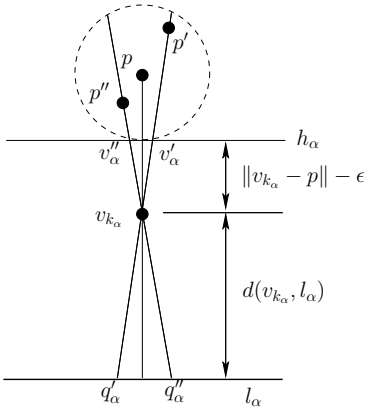

FIG. 2.3. *Definition of the lines l_α, h_α, and the points $q'_\alpha, q''_\alpha, v'_\alpha, v''_\alpha$.*

be the intersections between h_α and the lines $f_{\mathrm{pl}}(p', v_{k_\alpha})$ and $f_{\mathrm{pl}}(p'', v_{k_\alpha})$, respectively.

Now, $|A(v_{k_\alpha}, q'_\alpha, q''_\alpha)| = \frac{1}{2}\|q'_\alpha - q''_\alpha\| \operatorname{dist}(v_{k_\alpha}, l_\alpha)$. But from Figure 2.3, it is easy to see that $\|q'_\alpha - q''_\alpha\| = \frac{\operatorname{dist}(v_{k_\alpha}, l_\alpha)}{\|v_{k_\alpha} - p\| - \epsilon}\|v'_\alpha - v''_\alpha\|$ and that $\|v'_\alpha - v''_\alpha\| < \|p' - p''\|$. For

$K_\alpha(p) = \frac{1}{2}\frac{\mathrm{dist}(v_{k_\alpha}, l_\alpha)^2}{\|v_{k_\alpha} - p\| - \epsilon}$, the following is true:

$$|A(S(p')) - A(S(p''))| \le \sum_{\alpha=1}^{m} |A(v_{k_\alpha}, q'_\alpha, q''_\alpha)|$$
$$\le \sum_{\alpha=1}^{m} K_\alpha(p)\|p' - p''\|.$$

This fact is illustrated by Figure 2.4. This completes the proof that $Q \setminus \mathrm{Ve}_r(Q) \ni$

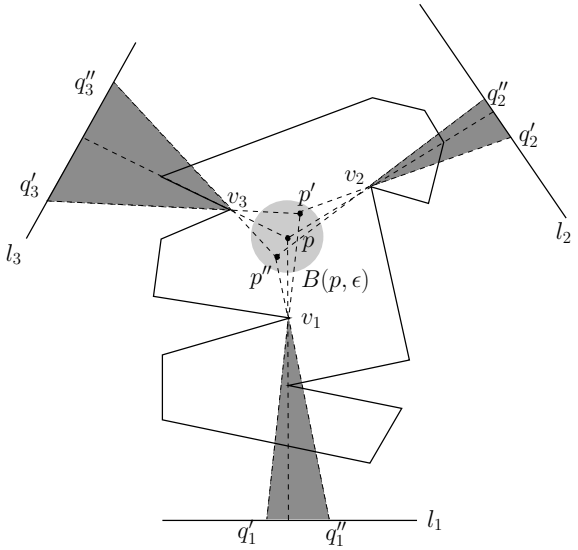

FIG. 2.4. *Upper bounds on the change in area. Here* $m = 3$.

$p \mapsto A \circ S(p)$ is locally Lipschitz. □

To obtain the expression for the generalized gradient of $A \circ S$, the polygon Q is partitioned as follows.

LEMMA 2.5. *Let* $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ *be the set of generalized inflection segments of* Q. *There exists a unique partition* $\{\overline{P}_\beta\}_{\beta\in\mathcal{B}}$ *of* Q *where* P_β *is a connected component of* $Q \setminus \bigcup_{\alpha\in A} I_\alpha$ *and* $\overline{P}_\beta$ *denotes its closure.*

Figure 2.5 illustrates this partition for the given nonconvex polygon. For $\beta \in \mathcal{B}$, define $A_\beta : \overline{P}_\beta \to \mathbb{R}_+$ by

$$A_\beta(p) = A \circ S(p), \quad \text{for } p \in P_\beta,$$

and by continuity on the boundary of P_β. It turns out that the maps A_β, $\beta \in \mathcal{B}$, are continuously differentiable* on $\overline{P}_\beta$. Equation (2.1) gives the value of the gradient for $p \in P_\beta$. However, in general, for $p \in \overline{P}_{\beta_1} \bigcap \ldots \bigcap \overline{P}_{\beta_m} \setminus \mathrm{Ve}_r(Q)$, based on Theorem 2.4 and Lemma 2.5, we can write

$$\partial(A \circ S)(p) = \mathrm{co}\left\{\mathrm{d}A_{\beta_1}(p), \ldots, \mathrm{d}A_{\beta_m}(p)\right\}. \tag{2.3}$$

*A function is continuously differentiable on a closed set if (1) it is continuously differentiable on the interior, and (2) the limit of the derivative at a point in the boundary does not depend on the direction from which the point is approached.

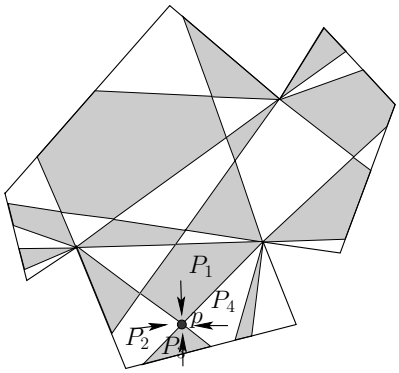

FIG. 2.5. *Partition of Q. The generalized gradient of the area function at p is the convex hull of the gradient of four functions $A_1, \ldots, A_4$ at p.*

This completes our study of the generalized gradient of the locally Lipschitz function $A \circ S$. Next, it is shown how this function is not regular in many interesting situations.

LEMMA 2.6. *There exists a nonconvex polygon Q such that the maps $A \circ S$ and $-A \circ S$ restricted to $Q \setminus \mathrm{Ve}_r(Q)$ are not regular.*

Proof. We present an example to justify the above statement. We refer the reader to Appendix A for the notion of right directional derivative and generalized directional derivative. In Figure 2.6, $\partial(A \circ S)(p') = \mathrm{co}\{\mathrm{d}A_1, \mathrm{d}A_2\}$ where $\|\mathrm{d}A_1\| \gg \|\mathrm{d}A_2\|$. Take a vector η' perpendicular to the generalized inflection segment to which p' belongs (see Figure 2.6). It is clear that $(A \circ S)'(p; \eta') = \mathrm{d}A_2 \cdot \eta'$. However, $(A \circ S)^0(p'; \eta') = \max\{\zeta \cdot \eta' | \zeta \in \partial(A \circ S)(p')\} = \mathrm{d}A_1 \cdot \eta' > \mathrm{d}A_2 \cdot \eta'$. Again, in

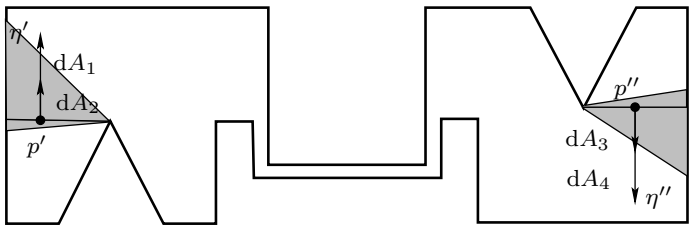

FIG. 2.6. *Example polygon for which $A \circ S$ and $-A \circ S$ restricted to $Q \setminus \mathrm{Ve}_r(Q)$ are not regular. Note here that $\mathrm{d}A_1$ and $\mathrm{d}A_2$ are not perfectly aligned with η'. Also, $\mathrm{d}A_3$ and $\mathrm{d}A_4$ are not perfectly aligned with η''.*

Figure 2.6, $\partial(-A \circ S)(p'') = \mathrm{co}\{-\mathrm{d}A_3, -\mathrm{d}A_4\}$, where $\| - \mathrm{d}A_4\| \gg \| - \mathrm{d}A_3\|$. Take a vector η'' perpendicular to the generalized inflection segment to which p'' belongs (see Figure 2.6). It is clear that $-(A \circ S)'(p''; \eta'') = -\mathrm{d}A_4 \cdot \eta''$. However, $(A \circ S)^0(p''; \eta'') = \max\{\zeta \cdot \eta'' | \zeta \in \partial(A \circ S)(p'')\} = -\mathrm{d}A_3 \cdot \eta'' > -\mathrm{d}A_4 \cdot \eta''$. □

3. An invariance principle in nonsmooth stability analysis. This section presents results on stability analysis for discontinuous vector fields via nonsmooth Lyapunov functions. The results extend the work in [2] and will be useful in the next control design section, see also [5]. We refer the reader to [8] and to the self-contained exposition in Appendix A for some useful nonsmooth analysis concepts.

In what follows we shall study differential equations of the form

$$\dot{x}(t) = X(x(t)),$$

where X is a discontinuous vector field on $\mathbb{R}^N$.

To analyze this differential equation we introduce a useful tool. Given a locally Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$, the *set-valued Lie derivative of f with respect to X* at x is defined as

$$\widetilde{\mathcal{L}}_X f(x) = \{a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ such that } \zeta \cdot v = a\,,\ \forall \zeta \in \partial f(x)\}\ .$$

For each $x \in \mathbb{R}^N$, $\widetilde{\mathcal{L}}_X f(x)$ is a closed and bounded interval in $\mathbb{R}$, possibly empty. If f is continuously differentiable at x, then $\widetilde{\mathcal{L}}_X f(x) = \{\mathrm{d}f \cdot v \mid v \in K[X](x)\}$. If, in addition, X is continuous at x, then $\widetilde{\mathcal{L}}_X f(x)$ corresponds to the singleton $\{\mathcal{L}_X f(x)\}$, the usual Lie derivative of f in the direction of X at x.

We are now ready to state the first result in this section.

LEMMA 3.1. *Let $X : \mathbb{R}^N \to \mathbb{R}^N$ be measurable and essentially locally bounded and let $f : \mathbb{R}^N \to \mathbb{R}$ be locally Lipschitz. Let $\gamma : [t_0, t_1] \to \mathbb{R}^N$ be a Filippov solution of X such that $f(\gamma(t))$ is regular for almost all $t \in [t_0, t_1]$. Then*

(i) $\frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t)))$ exists for almost all $t \in [t_0, t_1]$, and

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t))) \in \widetilde{\mathcal{L}}_X f(\gamma(t))$ for almost all $t \in [t_0, t_1]$.

Proof. The result is an immediate consequence of Lemma 1 in [2]. □

The following result is a generalization of the classic LaSalle Invariance Principle for smooth vector fields and smooth Lyapunov functions to the setting of discontinuous vector fields and nonsmooth Lyapunov functions.

THEOREM 3.2 (LaSalle Invariance Principle). *Let $X : \mathbb{R}^N \to \mathbb{R}^N$ be measurable and essentially locally bounded and let $S \subset \mathbb{R}^N$ be compact and strongly invariant for X. Let $C \subset S$ consist of a finite number of points and let $f : S \to \mathbb{R}$ be locally Lipschitz on $S \setminus C$ and bounded from below on S. Assume the following properties hold:*

(A1) if $x \in S \setminus C$, then either $\max \widetilde{\mathcal{L}}_X f(x) \leq 0$ or $\widetilde{\mathcal{L}}_X f(x) = \emptyset$,

(A2) if $x \in C$ and if γ is a Filippov solution of X with $\gamma(0) = x$, then $\lim_{t\to 0^-} f(\gamma(t)) \geq \lim_{t\to 0^+} f(\gamma(t))$, and

(A3) if $\gamma : \overline{\mathbb{R}}_+ \to S$ is a Filippov solution of X, then $f \circ \gamma$ is regular almost everywhere.

Define $Z_{X,f} = \left\{x \in S \setminus C \mid 0 \in \widetilde{\mathcal{L}}_X f(x)\right\}$ and let M be the largest weakly invariant set contained in $(\overline{Z}_{X,f} \cup C)$. Then the following statements hold:

(i) if $\gamma : \overline{\mathbb{R}}_+ \to S$ is a Filippov solution of X, then $f \circ \gamma$ is monotonically nonincreasing;

(ii) each Filippov solution of X with initial condition in S approaches M as $t \to +\infty$;

(iii) if M consists of a finite number of points, then each Filippov solution of X with initial condition in S converges to a point of M as $t \to +\infty$.

Proof. Fact (i) is a consequence of Assumptions (A1), (A2) and (A3), and of Lemma 3.1.

In what follows we shall require the following notion. Given a curve $\gamma : \mathbb{R}_+ \to \mathbb{R}^N$, the positive limit set of γ, denoted by $\Omega(\gamma)$, is the set of $y \in \mathbb{R}^N$ for which there exists a sequence $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $t_k < t_{k+1}$, for $k \in \mathbb{N}$, $\lim_{k\to+\infty} t_k = +\infty$, and $\lim_{k\to+\infty} \gamma(t_k) = y$. For $x \in S$, let γ_1 be a Filippov solution of X with $\gamma_1(0) = x$ and let $\Omega(\gamma_1)$ be the limit set of γ_1. Under this setting, $\Omega(\gamma_1)$ is nonempty, bounded,

connected and weakly invariant, see [8]. Furthermore, $\Omega(\gamma_1) \subset S$ because S is strongly invariant and closed.

To prove fact (ii), it suffices to show that $\Omega(\gamma_1) \subset \overline{Z}_{X,f} \cup C$. Trivially, $\Omega(\gamma_1) \cap C \subset C$. Let $y \in \Omega(\gamma_1) \backslash C$ so that f is locally Lipschitz at y. There exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} \gamma_1(t_k) = y$. Because $f \circ \gamma_1$ is monotonically nonincreasing and f is bounded from below, $\lim_{t\to+\infty} f(\gamma_1(t))$ exists and is equal to, say, $a \in \mathbb{R}$. Now, by continuity of f, $a = \lim_{k\to+\infty} f \circ \gamma_1(t_k) = f(y)$. This proves that $f(y) = a$ for all $y \in \Omega(\gamma_1) \setminus C$. At this point we distinguish two cases. First, assume that y is an isolated point in $\Omega(\gamma_1)$. Then clearly, there exists a Filippov solution of X, say γ_2, such that $\gamma_2(t) = y$ for all $t \geq 0$. Hence $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_2(t)) = 0$, and, by Lemma 3.1, $0 \in \widetilde{\mathcal{L}}_X f(\gamma_2(t))$ or in other words $y \in Z_{X,f}$. Second, assume that y is not isolated in $\Omega(\gamma_1)$, and let γ_2 be a Filippov solution of X with $\gamma_2(0) = y$. Since f is continuous at y and $\Omega(\gamma_1)$ contains a finite number of points of discontinuity of f, there exists $\delta > 0$ such that $f(y') = a$ for all $y' \in B(y, \delta) \cap \Omega(\gamma_1)$. Therefore, there exists $t' > 0$ such that $f(\gamma_2(t)) = a$ for all $t \in [0, t']$. Hence, we have $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_2(t)) = 0$ for all $t \in [0, t']$. It follows from Lemma 3.1 that for all $t \in [0, t']$, we have $0 \in \widetilde{\mathcal{L}}_X f(\gamma_2(t))$ or in other words $\gamma_2(t) \in Z_{X,f}$. By continuity of γ_2 at $t = 0$, we have that $\gamma_2(0) = y \in \overline{Z}_{X,f}$. Since $\Omega(\gamma_1)$ is weakly invariant, we have $\Omega(\gamma_1) \subset M$ and hence γ_2 approaches M.

We now prove fact (iii). If M consists of a finite number of points, and since $\Omega(\gamma_1) \subset M$ is connected, $\Omega(\gamma_1)$ is a point. Hence, by the argument in the preceding paragraph, each Filippov solution of X approaches a point of M. In other words, it converges to a point of M. □

COROLLARY 3.3. *The LaSalle Invariance Principle is valid under the following relaxed assumption:*

(A3) if $\gamma : \overline{\mathbb{R}}_+ \to S$ is a Filippov solution of X, then almost everywhere either $f \circ \gamma$ or $-f \circ \gamma$ is regular.

Proof. The proof is a consequence of the fact that $\frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t)))$ exists and belongs to $\widetilde{\mathcal{L}}_X f(\gamma(t))$ if and only if $\frac{\mathrm{d}}{\mathrm{d}t}(-f(\gamma(t)))$ exists and belongs to $\widetilde{\mathcal{L}}_X(-f)(\gamma(t))$. Thus result (ii) of Lemma 3.1 still holds and the proof of the LaSalle Invariance Principle remains unchanged. □

4. Maximizing the area visible from a mobile observer. In this section we build on the analysis results obtained thus far to design an algorithm that maximizes the area visible to a mobile observer. We aim to reach local maxima of the visible area $A \circ S$ by designing some appropriate form of a gradient flow for the discontinuous function $A \circ S$. We now present an *introductory and incomplete* version of the algorithm: the objective is to steer the mobile observer along a path for which the visible area is guaranteed to be nondecreasing.

Name:	Increase visible area for Q
Goal:	Maximize the area visible to a mobile observer
Assumption:	Generalized inflection segments of Q do not intersect. Initial position does not belong to a generalized inflection segment.

Let $p(t)$ denote the observer position at time t inside the nonconvex polygon Q. The observer performs the following tasks at each time instant:

compute visibility polygon $S(p(t)) \subset Q$,
if $p(t)$ does not belong to any generalized inflection segment or to the boundary of Q **then**
move along the versor of the gradient $\mathrm{d}(A \circ S)$
else if $p(t)$ belongs to a generalized inflection segment but not to the boundary of Q **then**
depending on the generalized gradient $\partial(A \circ S)$, either slide along the segment or leave the segment in an appropriate direction
else if $p(t)$ belongs to the boundary of Q but not to a reflex vertex, **then**
depending on the projection of $\partial(A \circ S)$ along the boundary, either slide along the boundary or move in an appropriate direction toward the interior of Q
else
either follow a direction of ascent of $A \circ S$ or stop
end if

The remainder of this section is dedicated to formalizing this loose description.

4.1. A modified gradient vector field. Before describing the algorithm to maximize the area visible to the mobile observer, we introduce the following useful notions. Given a simple polygon Q with $\mathrm{Ve}(Q) = (v_1, \ldots, v_n)$ and $\epsilon > 0$, define the following quantities:

(i) let the *ϵ-expansion of Q* be $Q^\epsilon = \{p \mid ||p - q|| \leq \epsilon \text{ for some } q \in Q\}$,
(ii) for $i \in \{1, \ldots, n\}$, let P_i^ϵ be the open set delimited by the edge $\overline{v_i v_{i+1}}$, the bisectors of the external angles at v_i and v_{i+1} and the boundary of Q^ϵ,
(iii) for ϵ small enough and for any point p in Q^ϵ, let $\mathrm{prj}_Q(p)$ be uniquely equal to $\arg\min\{||p' - p|| \mid p' \in \partial Q\}$, and
(iv) for $p \in \cup_{i \in \{1,\ldots,n\}} P_i^\epsilon$, let the *outward normal* $n(\mathrm{prj}_Q(p))$ be the unit vector directed from $\mathrm{prj}_Q(p)$ to p.

We illustrate these notions in Figure 4.1. Note that $\mathrm{prj}_Q(p)$ can never be a reflex vertex. We can now define a vector field on Q^ϵ as follows:

$$X_Q(p) = \begin{cases} \mathrm{vers}(\mathrm{d}(A \circ S)(p)), & \text{if } p \in \mathring{Q} \setminus \{I_\alpha\}_{\alpha \in \mathcal{A}}, \\ -n(\mathrm{prj}_Q(p)), & \text{if } p \in P_i^\epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

(Recall that the versor operator is defined by $\mathrm{vers}(Y) = Y/\|Y\|$ if $Y \in \mathbb{R}^2 \setminus \{0\}$ and by $\mathrm{vers}(0) = 0$.) Note that X_Q is well-defined because at $p \in \mathring{Q} \setminus \{I_\alpha\}_{\alpha \in \mathcal{A}}$ the function $A \circ S$ is analytic. Clearly, X_Q is not continuous on Q^ϵ. However, the set of points where it is discontinuous is of measure zero. Almost everywhere in the interior of Q, the vector field X_Q is equal to the normalized gradient of $A \circ S$ as depicted in Figure 2.2.

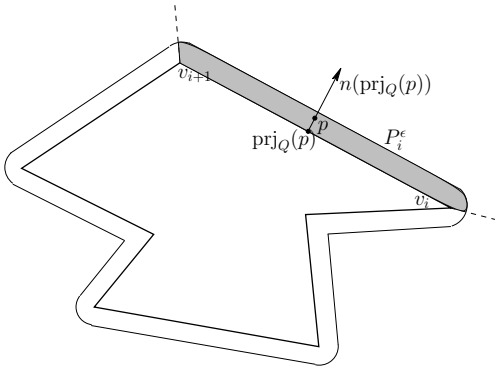

FIG. 4.1. *The ϵ-expansion Q^ϵ of the simple polygon Q, an open set P_i^ϵ and the corresponding outward normal $n(\mathrm{prj}_Q(p))$.*

REMARK 4.1. An important observation in this setting is that at all points p where $A \circ S$ is locally Lipschitz, we have $K[\mathrm{d}(A \circ S)](p) = \partial(A \circ S)(p)$. In such a case it is also true that for all $\eta \in \partial(A \circ S)(p)$, there exists at least one $\delta > 0$ such that $\delta\eta \in K[X_Q](p)$ and vice versa.

We now present the differential equation describing the motion of the observer:

$$\dot{p}(t) = X_Q(p(t)). \tag{4.1}$$

A Filippov solution of (4.1) on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as a solution of the differential inclusion

$$\dot{p}(t) \in K[X_Q](p(t)), \tag{4.2}$$

where $K[X_Q]$ is the usual Filippov differential inclusion associated with X_Q, see Appendix A. Since X_Q is measurable and bounded, the existence of a Filippov solution is guaranteed. We study uniqueness and completeness of Filippov solutions in the following lemma.

LEMMA 4.2. *The following statements hold true:*

(i) there exists a simple polygon Q for which the corresponding vector field X_Q admits multiple Filippov solutions;

(ii) any simple polygon Q is a strongly invariant set for the corresponding vector field X_Q and, therefore, any Filippov solution is defined over $\overline{\mathbb{R}}_+$.

Proof. We present an example to justify the statement (i). In Figure 4.2, at the point p_0 on the generalized inflection segment, both directions η_1 and η_2 belong to $\partial(A \circ S)(p_0)$. Three distinct Filippov solutions of equation (4.1) exist. Two of the solutions start from p_0 along the two directions η_1 and η_2 while the third solution is $p(t) = p_0$ for all $t \geq 0$. Statement (ii) is a consequence of the definition of X_Q on P_i^ϵ

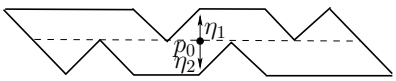

FIG. 4.2. *Three Filippov solutions exist starting from the point p_0.*

for $i \in \{1, \ldots, n\}$. □

We now claim that any solution of the differential inclusion (4.2) has the property that the visible area increases monotonically. To prove these desirable properties, we first present the following results in nonsmooth analysis.

4.2. Properties of solutions and convergence analysis. To prove the convergence properties of the solution of (4.2) using the results presented in Section 3, we must first define a suitable Lyapunov function. Intuitively since our objective is to maximize the visible area, our Lyapunov function should be closely related to it. For $\epsilon > 0$, we now define the *extended area function* A_Q^ϵ at all points $p \in Q \bigcup \{\cup_i P_i^\epsilon\}$. The extended function coincides with the original function on the interior and on the boundary of Q and is defined appropriately outside:

$$A_Q^\epsilon(p) = \begin{cases} A \circ S(p), & \text{if } p \in Q, \\ A \circ S(\text{prj}_Q(p)) - ||p - \text{prj}_Q(p)||, & \text{if } p \in \cup_i P_i^\epsilon. \end{cases}$$

For all $p \in \partial Q \setminus \text{Ve}\, Q$, A_Q^ϵ satisfies (see Figure 4.3):

$$A_Q^\epsilon{}'(p; n(\text{prj}_Q(p))) = -1.$$

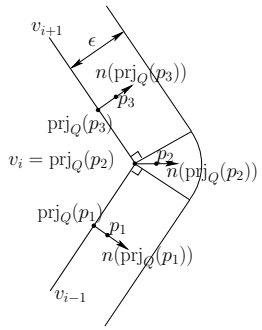

FIG. 4.3. *Extending the function $A \circ S$ to A_Q^ϵ. Note the direction of $n(\text{prj}_Q(p_i))$ at all points p_i.*

REMARK 4.3. The extended area function A_Q^ϵ is locally Lipschitz on $(Q \setminus \text{Ve}_r(Q)) \bigcup \{\cup_i P_i^\epsilon\}$ and analytic almost everywhere on $Q \bigcup \{\cup_i P_i^\epsilon\}$.

The following theorem is important to prove that such a function leads to a monotonically nondecreasing value of the area of the visibility polygon.

THEOREM 4.4. *Let $G(Q)$ be the subset of Q where both maps $p \mapsto -A_Q^\epsilon(p)$ and $p \mapsto A_Q^\epsilon(p)$ are not regular. Then any Filippov solution $\gamma : \overline{\mathbb{R}}_+ \to Q$ of X_Q has the property that $\gamma(t) \notin G(Q)$ for almost all $t \in \overline{\mathbb{R}}_+$ unless γ reaches a critical point of $K[X_Q]$.*

Proof. Note that $G(Q)$ is a subset of $\cup_{\alpha \in \mathcal{A}} I_\alpha$. This is a consequence of Theorem 2.2 and the fact that functions are regular at points of differentiability. Given a generalized inflection segment I_α, let l_α be the line extending I_α and let t_α be one of the two unit tangent vectors to I_α. A Filippov solution γ of X_Q slides along I_α starting from $p_0 \in I_\alpha$ only if $\partial A_Q^\epsilon(p_0)$ contains either t_α or $-t_\alpha$. It then suffices to show that if $\partial A_Q^\epsilon(p_0)$ contains t_α or $-t_\alpha$, then either A_Q^ϵ or $-A_Q^\epsilon$ is regular at p_0. Let us also assume that p_0 does not belong to any other generalized inflection segment. If this were not the case, then either p_0 is a critical point or the Filippov solution does not belong to the point of intersection for almost all $t \in \overline{\mathbb{R}}_+$.

Let l_α divide $\mathbb{R}^2$ into two open half planes H_1 and H_2. There exists $\delta > 0$ such that A_Q^ϵ is analytic on $H_i \cap B(p_0, \delta)$, $i \in \{1, 2\}$, see Figure 4.4. On l_α, we have $(A_Q^\epsilon)_1 = (A_Q^\epsilon)_2$ where $(A_Q^\epsilon)_i$ is the function A_Q^ϵ restricted to H_i. Let $p' \in B(p_0, \delta)$ and, without

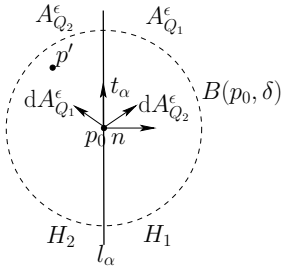

FIG. 4.4. *The point p_0 lies on the generalized inflection segment l. H_1 and H_2 are half planes on either side of l. n and t are normal and parallel to l respectively. The other arrows indicate the directions of $\mathrm{d}A^\epsilon{}_Q(p)$ on either side of l.*

loss of generality, let $p' \in H_2$. Let n be the normal to I_α at p_0 pointing away from p'. Note that in terms of the notation introduced in Section 4.1, $n = -n(\mathrm{prj}_{l_\alpha}(p'))$ where $\mathrm{prj}_{l_\alpha}(p') = \arg\min\{||p' - p|| \mid p \in l_\alpha\}$. Now, $(A^\epsilon_Q)_1$ can be extended to $H_2 \cap B(p_0, \delta)$ by analyticity. Likewise, $(A^\epsilon_Q)_2$ can be extended to $H_1 \cap B(p_0, \delta)$. Since the functions $(A^\epsilon_Q)_i$, $i \in \{1, 2\}$, are analytic, they can be written as the expansions of their Taylor series':

$$(A^\epsilon_Q)_i(p') = (A^\epsilon_Q)_i(p_0) + \mathrm{d}((A^\epsilon_Q)_i)(p_0) \cdot (p' - p_0) + O(\|p' - p_0\|^2).$$

It follows from the above set of equations that:

$$(A^\epsilon_Q)_2(p') - (A^\epsilon_Q)_1(p') = \big(\mathrm{d}((A^\epsilon_Q)_2) - \mathrm{d}((A^\epsilon_Q)_1)\big) \cdot (p' - p_0) + O(\|p' - p_0\|^2).$$

Note that n is the same for all $p' \in H_2$. Now, $p' - p_0 = -c_1 n + c_2 t_\alpha$ such that $c_1 \geq 0$. Also, $\mathrm{d}(A^\epsilon_Q)_1(p_0) \cdot t_\alpha = \mathrm{d}(A^\epsilon_Q)_2(p_0) \cdot t_\alpha$ since $(A^\epsilon_Q)_1(p) = (A^\epsilon_Q)_2(p)$ for $p \in I_\alpha$. Therefore,

$$(A^\epsilon_Q)_2(p') - (A^\epsilon_Q)_1(p') = c_1 \left(\mathrm{d}(A^\epsilon_Q)_1(p_0) \cdot n - \mathrm{d}(A^\epsilon_Q)_2(p_0) \cdot n\right) + O(\|p' - p_0\|^2).$$

Now, either t_α or $-t_\alpha$ belongs to $\partial A^\epsilon_Q(p_0) = \mathrm{co}\{\mathrm{d}(A^\epsilon_Q)_1, \mathrm{d}(A^\epsilon_Q)_2\}$ if and only if the product of $\mathrm{d}(A^\epsilon_Q)_1(p_0) \cdot n$ and $\mathrm{d}(A^\epsilon_Q)_2(p_0) \cdot n$ is less than or equal to zero (see Figure 4.4). If $\mathrm{d}(A^\epsilon_Q)_1(p_0) \cdot n = 0$ and $\mathrm{d}(A^\epsilon_Q)_2(p_0) \cdot n = 0$, then clearly A^ϵ_Q is C^1 at p_0 and hence regular. Otherwise, let us assume, without loss of generality, that $\mathrm{d}(A^\epsilon_Q)_1(p_0) \cdot n - \mathrm{d}(A^\epsilon_Q)_2(p_0) \cdot n < 0$. Therefore, there exists $\eta_2 > 0$ such that $(A^\epsilon_Q)_2(p') - (A^\epsilon_Q)_1(p') \leq 0$ for $p' \in H_2 \cap B(p_0, \eta_2)$. Similarly, there exists $\eta_1 > 0$ such that for $p' \in H_1 \cap B(p_0, \eta_1)$, we have $(A^\epsilon_Q)_1(p') - (A^\epsilon_Q)_2(p') \leq 0$. Thus, there exists a neighborhood around p_0 where $A^\epsilon_Q(p) = \min\{(A^\epsilon_Q)_1(p), (A^\epsilon_Q)_2(p)\}$ or $-A^\epsilon_Q(p) = \max\{-(A^\epsilon_Q)_1(p), -(A^\epsilon_Q)_2(p)\}$. Since $(A^\epsilon_Q)_i$, $i \in \{1, 2\}$, are smooth functions, it follows from Proposition 2.3.12 in [4] that $-A^\epsilon_Q$ is regular at p_0. On the other hand, if we assume that $\mathrm{d}(A^\epsilon_Q)_1(p_0) \cdot n - \mathrm{d}(A^\epsilon_Q)_2(p_0) \cdot n > 0$, then we get that A^ϵ_Q is regular at p_0. □

In the following theorem, the functions A^ϵ_Q and $-A^\epsilon_Q$ are used as candidate Lyapunov functions to show the convergence properties of Filippov solutions of X_Q.

THEOREM 4.5. *Any Filippov solution $\gamma : \overline{\mathbb{R}}_+ \to Q$ of X_Q has the following properties:*

(i) $t \mapsto A \circ S(\gamma(t))$ is continuous and monotonically nondecreasing,

(ii) γ approaches the set of critical points of $K[X_Q]$.

Proof. Let us start by showing that, if γ is a Filippov solution of X_Q, then $A \circ S \circ \gamma$ is continuous. The reader is referred to Figure 4.5 for an introduction of

notations used. Let X_Q^r and X_Q^θ be the components of X_Q parallel and perpendicular to $p-v$ respectively. Similarly, let $\mathrm{d}(A\circ S(p))^r$ and $\mathrm{d}(A\circ S(p))^\theta$ be the components of $\mathrm{d}(A\circ S(p))$ parallel and perpendicular to $p-v$ respectively. Note that if $\|\mathrm{d}(A\circ S(p))\| \neq 0$, then $\|X_Q^r\| = \frac{\|\mathrm{d}(A\circ S(p))^r\|}{(\|\mathrm{d}(A\circ S(p))^r\|^2+\|\mathrm{d}(A\circ S(p))^\theta\|^2)^{1/2}}$ and $\|X_Q^\theta\| = \frac{\|\mathrm{d}(A\circ S(p))^\theta\|}{(\|\mathrm{d}(A\circ S(p))^r\|^2+\|\mathrm{d}(A\circ S(p))^\theta\|^2)^{1/2}}$. Let $\epsilon > 0$ be such that $\|\mathrm{d}(A\circ S(p))\| \neq 0$ for all $p \in B(v,\epsilon)\cap D$. For now, let us also assume that $\{\cup_{\alpha\in\mathcal{A}} I_\alpha\}\cap B(v,\epsilon)\cap D = \emptyset$. We now claim that in $B(v,\epsilon)\cap D$, $\mathrm{d}(A\circ S(p))^\theta = \Omega(1/\|p-v\|)$ and $\mathrm{d}(A\circ S(p))^r = O(1)$. Notice that $\mathrm{d}(A\circ S(p)) = \mathrm{d}(A\circ S(p))^r + \mathrm{d}(A\circ S(p))^\theta = \sum_i \frac{\partial(A\circ S)}{\partial u_i}\mathrm{d}u_i(p)$. Let $u_1 = u$. From (2.2), it is clear that $\frac{\partial(A\circ S)}{\partial u}\mathrm{d}u(p)$ is perpendicular to $p-v$ and hence contributes only to $\mathrm{d}(A\circ S(p))^\theta$. Also $\|\sum_{i\geq 2}\frac{\partial(A\circ S)}{\partial u_i}\mathrm{d}u_i(p)\|$ is bounded for all $p\in B(v,\epsilon)\cap D$. Therefore, $\mathrm{d}(A\circ S(p))^\theta = \frac{\partial(A\circ S)}{\partial u}\mathrm{d}u(p) + \Omega(1) = \Omega(\|\frac{\partial(A\circ S)}{\partial u}\mathrm{d}u(p)\|)$ and $\mathrm{d}(A\circ S(p))^r = O(1)$. Again from (2.2), we have

$$\|\frac{\partial(A\circ S)}{\partial u}\mathrm{d}u(p)\| = \frac{\mathrm{dist}(v,l)}{2}\frac{\|p-v\|\,|\,\mathrm{dist}(u_2,l)-\mathrm{dist}(u_n,l)|}{(\mathrm{dist}(p,l)-\mathrm{dist}(v,l))^2}.$$

Now, $|\,\mathrm{dist}(p,l)-\mathrm{dist}(v,l)| \leq \|p-v\|$. Therefore,

$$\|\frac{\partial(A\circ S)}{\partial u}\mathrm{d}u(p)\| = \Omega\left(\frac{|\,\mathrm{dist}(u_2,l)-\mathrm{dist}(u_n,l)|}{\|p-v\|}\right).$$

Since p does not lie on a generalized inflection segment, either $u_n = v$ or $u_2 = v$. Without loss of generality, let $u_n = v$. Since u belongs to l, clearly u_2 must belong to l. Hence $|\,\mathrm{dist}(u_2,l)-\mathrm{dist}(u_n,l)| = \mathrm{dist}(v,l)$ and is a constant for all $p\in B(v,\epsilon)\cap D$. Thus

$$\|\frac{\partial(A\circ S)}{\partial u}\mathrm{d}u(p)\| = \Omega\left(\frac{1}{\|p-v\|}\right).$$

Hence, $\mathrm{d}(A\circ S(p))^\theta = \Omega(\frac{1}{\|p-v\|})$. In other words there exist constants $k_r > 0$ and $k_\theta > 0$ such that $\|\mathrm{d}(A\circ S(p))^r\| \leq k_r$ and $\|\mathrm{d}(A\circ S(p))^\theta\| \geq \frac{k_\theta}{\|p-v\|}$. Therefore $\frac{\|\mathrm{d}(A\circ S(p))^\theta\|}{\|\mathrm{d}(A\circ S(p))^r\|} \geq \frac{k_\theta}{k_r\|p-v\|}$. It follows that

$$\begin{aligned}\|X_Q^r\| &= \frac{1}{(1+\frac{\|\mathrm{d}(A\circ S(p))^\theta\|^2}{\|\mathrm{d}(A\circ S(p))^r\|^2})^{1/2}} \leq \frac{1}{(1+\frac{k_\theta^2}{k_r^2\|p-v\|^2})^{1/2}} = \frac{k_r\|p-v\|}{(k_\theta^2+k_r^2\|p-v\|^2)^{1/2}}\\ &\leq \frac{k_r\|p-v\|}{k_\theta}.\end{aligned}$$

Note that a convex combination of finitely many X_Q^r will also admit a similar inequality and so the assumption that $\{\cup_{\alpha\in\mathcal{A}} I_\alpha\}\cap B(v,\epsilon)\cap D = \emptyset$ is not limiting. Now let $\gamma(t)$ be a solution of X_Q such that $\gamma(0) = v$. Let T be any time such that $\|\gamma(T)-v\| = R$ and for all $t\in[0,T]$, $\gamma(t)\in B(v,\epsilon)\cap D$ and $X_Q^r(\gamma(t))$ is directed away from v. Then clearly, $R = \int_0^T X_Q^r\mathrm{d}t \leq R\frac{k_r}{k_\theta}T$. In other words the time T taken for a trajectory to travel any distance R is greater than $\frac{k_\theta}{k_r}$. This is a contradiction. Therefore, our assumption that for all $t\in[0,T]$, $\gamma(t)\in B(v,\epsilon)\cap D$ is false. So, the trajectory must belong to C for some finite time interval contained in $[0,T]$. We can choose R as small as possible and this implies that there exists a finite time interval $[0,T_C]$ for which $\gamma(t)\in C$. It follows trivially that $t\mapsto A\circ S(\gamma(t))$ is right continuous

at t where $\gamma(t) = v$. We can prove similarly that $t \mapsto A \circ S(\gamma(t))$ is left continuous at t where $\gamma(t) = v$ by considering the vector field $-X_Q$ in place of X_Q. This completes the proof that $t \mapsto A \circ S(\gamma(t))$ is continuous.

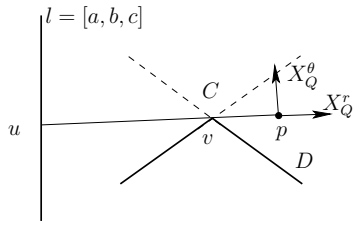

FIG. 4.5. *Illustration of various notions used in Theorem 4.5. The dashed lines represent generalized inflection segments generated by the reflex vertex v and vertices adjacent to it. These divide the region around v that is inside Q into three subregions C , D and E. $u \in \mathrm{Ve}(S(p))$ lies on the line l. The generalized inflection segments including the vertex v are assumed to belong to region C. Note that $D \cap C = \emptyset$.*

Next we show that Assumptions (A1), (A2) and (A3) in Theorem 3.2 hold. Let $p \in Q \setminus \mathrm{Ve}_r(Q)$ and take $a \in \widetilde{\mathcal{L}}_{X_Q}(-A_Q^\epsilon)(p)$. By definition, there exists $k \in K[X_Q](p)$ such that $a = k \cdot \zeta$ for all $\zeta \in -\partial A_Q^\epsilon(p)$. In particular, it is true for $\zeta = -\delta k$, for some $\delta > 0$, see Remark 4.1. Therefore, $a = -\delta \|k\|^2 \leq 0$. This proves that either $\max \widetilde{\mathcal{L}}_{X_Q}(-A_Q^\epsilon)(p) \leq 0$ or $\widetilde{\mathcal{L}}_{X_Q}(-A_Q^\epsilon)(p) = \emptyset$, i.e., Assumption (A1) is satisfied. Assumption (A2) is a consequence of the continuity of $A \circ S \circ \gamma$. Finally, Assumption (A3) is a consequence of Theorem 4.4. Applying now Theorem 3.2 and its corollary, we conclude that fact (i) holds. Moreover, we also deduce that any Filippov solution of X_Q converges to the largest weakly invariant set M contained in $\overline{Z}_{X_Q,-A_Q^\epsilon} \cup \mathrm{Ve}_r(Q)$.

To prove fact (ii), let us show that $M = \{p \in Q \mid 0 \in K[X_Q](p)\} \cap (\overline{Z}_{X_Q,-A_Q^\epsilon} \cup \mathrm{Ve}_r(Q))$. Based on Theorem 4.4, Theorem 3.2 and Corollary 3.3, it suffices to show that M is contained in $\{p \in Q \mid 0 \in K[X_Q](p)\}$. Let us note that the set $\{p \in Q \mid 0 \in K[X_Q](p)\}$ is weakly invariant and can be established to be closed following the same reasoning as in Proposition 2.1.1 in [5]. Let $x \in Z_{X_Q,-A_Q^\epsilon}$. Then, $0 \in \widetilde{\mathcal{L}}_{X_Q}(-A_Q^\epsilon)(x)$, i.e., there exists $k \in K[X_Q](x)$ such that $\zeta \cdot k = 0$ for all $\zeta \in -\partial A_Q^\epsilon(x)$. But, $k \in K[X_Q](x)$ implies that there exists $\delta > 0$ such that $\delta k \in -\partial A_Q^\epsilon(x)$, see Remark 4.1. Thus, for $\zeta = \delta k$, we get $\delta \|k\|^2 = 0$, that is, $0 \in K[X_Q](x)$. This shows that $Z_{X_Q,-A_Q^\epsilon} \subset \{p \in Q \mid 0 \in K[X_Q](x)\}$. Next, let $x \in \mathrm{Ve}_r(Q) \cap M$. If the set $\{x\}$ is weakly invariant, then by definition $0 \in K[X_Q](x)$. If on the other hand x is not isolated in M, then there exists a sequence of points $\{x_m\}_{m \in \mathbb{N}}$ converging to x such that $x_m \in Z_{X_Q,-A_Q^\epsilon}$ or, alternatively, $0 \in K[X_Q](x_m)$. Because $\{p \in Q \mid 0 \in K[X_Q](p)\}$ is closed, it follows that $0 \in K[X_Q](x)$. Thus we proved that any weakly invariant set contained in $Z_{X_Q,-A_Q^\epsilon} \cup \mathrm{Ve}_r(Q)$ is a subset of $\{p \in Q \mid 0 \in K[X_Q](p)\}$. Again, as in Proposition 2.1.1 in [5], it can be shown that $Z_{X_Q,-A_Q^\epsilon}$ is a closed set and hence the claim that $M \subset \{p \in Q \mid 0 \in K[X_Q](p)\}$ follows. □

Theorem 4.5 implies that the single observer converges to a critical point of $A \circ S$ or to a reflex vertex of Q. However, as shown in Figure 5.2, the presence of noise or computational inaccuracies actually works to drive the observer away from a reflex vertex that is not a local maximum. This will also be true for other critical points that are not local maxima.

5. Simulation results. To conduct experiments, a simulation environment has been developed in Matlab®. There are two levels of the code. The lower level consists of a library containing routines to answer queries such as whether two points in a two dimensional polygonal environment are visible to each other. The higher level utilizes these routines and consists of two major portions. In the first, the vertices of the visibility polygon are obtained by means of an $O(n^2)$ algorithm, where n is the number of vertices of the polygonal environment. These are then sorted in counterclockwise order to compute the visibility polygon. The second consists of the controller which decides the direction and the step size of the observer motion at each time instant. The main task of the controller is the calculation of the generalized gradient of the visible area function which is a natural outcome of (2.1) and (2.3). *Such a framework gives the flexibility to easily implement other visibility based algorithms for single or multiple observers in a polygonal environment. This can be done by extracting the appropriate information using the low level functions and implementing the desired controller.*

Figures 5.2 and 5.4 illustrate the performance of the gradient algorithm in equation (4.2). Computational inaccuracies in the implementation of the algorithm to calculate the visibility polygon have been noticed in some configurations; see the plot of the evolution of visible area with time in Figure 5.2. See Figure 5.3 for the phase portrait of the vector field X_Q for the polygon in Figure 5.1. Simulation results for an observer in a similar polygonal environment containing a hole is shown in Figure 5.5. Our experiments suggest that the observer reaches a local maximum of the visible area in finite time, however this can be shown not to be true in general.

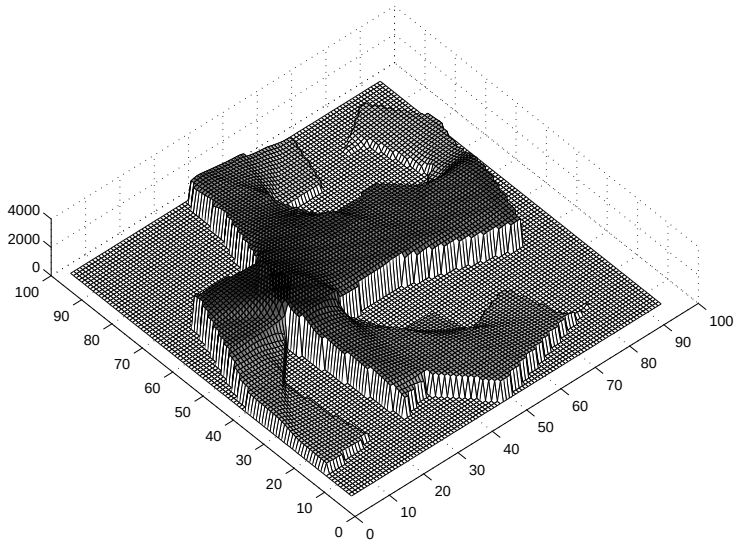

FIG. 5.1. *Example of visible area function over a typical nonconvex polygon.*

6. Conclusions. This paper introduces a gradient-based algorithm to optimally locate a mobile observer in a nonconvex environment. We have presented nonsmooth analysis and control design results. The simulation results illustrate that, in the presence of noise, the observer reaches a local maximum of the visible area. In an "highly nonconvex" environment, a single observer may not be able to see a large

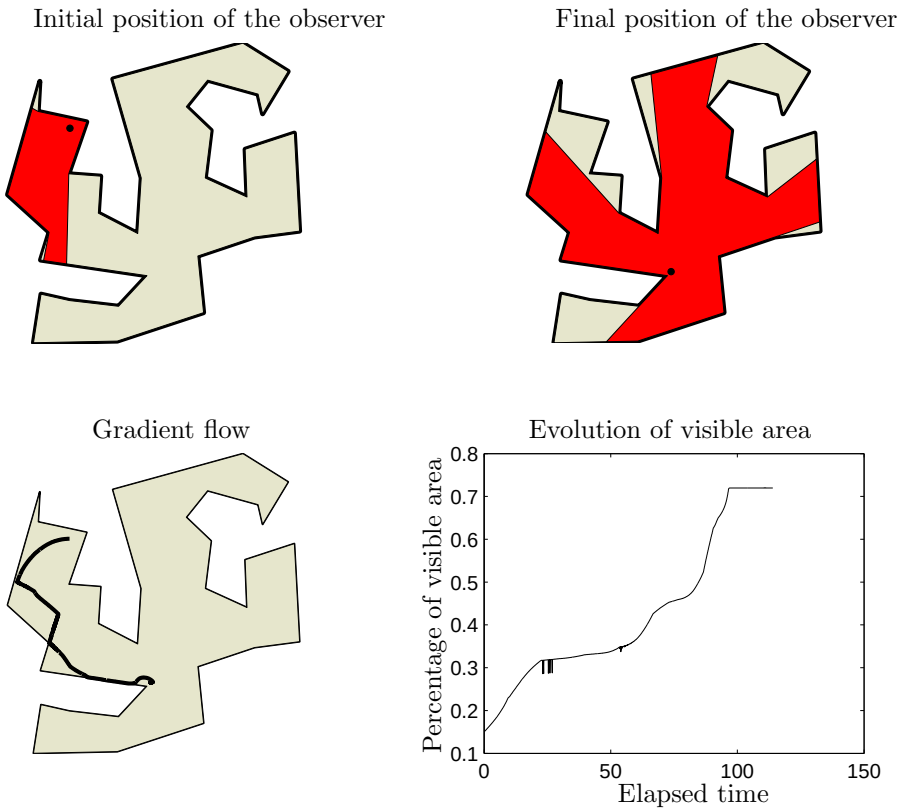

FIG. 5.2. *Simulation results of the gradient algorithm for the nonconvex polygon depicted in Figure 5.1. The observer arrives, in finite time, at a local maximum. Note here that the observer visits a reflex vertex at some point in its trajectory but comes out of it due to computational inaccuracies because it is not a local maximum.*

fraction of the environment. In such a case, a team of observers can be deployed to achieve the same task. We therefore plan to investigate this same visibility objective for teams of observers. Other directions of future research include practical robotic implementation issues as well as other combined mobility and visibility problems.

REFERENCES

[1] P. K. AGARWAL AND M. SHARIR, *Efficient algorithms for geometric optimization*, ACM Computing Surveys, 30 (1998), pp. 412–458.

[2] A. BACCIOTTI AND F. CERAGIOLI, *Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions*, ESAIM. Control, Optimisation & Calculus of Variations, 4 (1999), pp. 361–376.

[3] O. CHEONG, A. EFRAT, AND S. HAR-PELED, *On finding a guard that sees most and a shop that sells most*, in ACM-SIAM Symposium on Discrete Algorithms, New Orleans, LA, Jan. 2004, pp. 1091–1100.

[4] F. H. CLARKE, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, 1983.

[5] J. CORTÉS AND F. BULLO, *Coordination and geometric optimization via distributed dynamical systems*, SIAM Journal on Control and Optimization, (2004). To appear.

[6] M. DE BERG, M. VAN KREVELD, AND M. OVERMARS, *Computational Geometry: Algorithms and Applications*, Springer Verlag, New York, 1997.

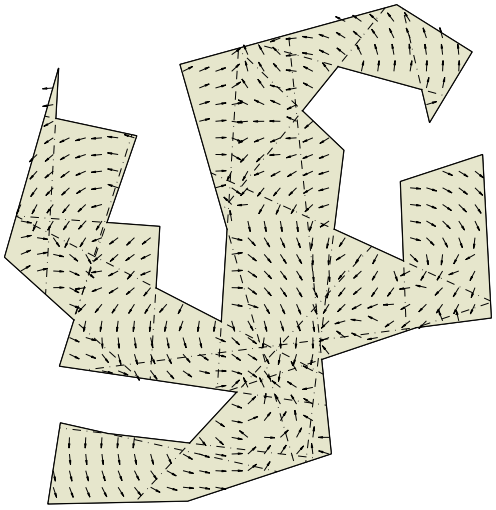

FIG. 5.3. *Example of vector field over the nonconvex polygon in Figure 5.1.*

[7] P. K. EASON AND J. A. STAMPS, *The effect of visibility on space use by territorial red-capped cardinals*, Behaviour, 138 (2001), pp. 19–30.

[8] A. F. FILIPPOV, *Differential Equations with Discontinuous Righthand Sides*, vol. 18 of Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.

[9] H. H. GONZÁLEZ-BAÑOS AND J.-C. LATOMBE, *Navigation strategies for exploring indoor environments*, International Journal of Robotics Research, 21 (2002), pp. 829–848.

[10] J. E. GOODMAN. AND R. POLLACK, *Foundations of a theory of convexity on affine Grassmann manifolds*, Mathematika. A Journal of Pure and Applied Mathematics, 42 (1995), pp. 305–328.

[11] D. S. HOCHBAUM AND A. PATHRIA, *Analysis of the greedy approach in problems of maximum k-coverage*, Naval Research Logistics. An International Journal, 45 (1998), pp. 615–627.

[12] K. KAKUSHO, T. KITAHASHI, K. KONDO, AND J.-C. LATOMBE, *Continuous purposive sensing for 2D map building*, in Proceedings of the IEEE International Conference of Systems, Man and Cybernetics, Vancouver, BC, Canada, 1995, pp. 1472–1477.

[13] B. PADEN AND S. S. SASTRY, *A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators*, IEEE Transactions on Circuits and Systems, 34 (1987), pp. 73–82.

[14] T. C. SHERMER, *Recent results in art galleries*, IEEE Proceedings, 80 (1992), pp. 1384–1399.

[15] D. SHEVITZ AND B. PADEN, *Lyapunov stability theory of nonsmooth systems*, IEEE Transactions on Automatic Control, 39 (1994), pp. 1910–1914.

[16] P. VALTR, *Guarding galleries where no point sees a small area*, Israel Journal of Mathematics, 104 (1998), pp. 1–16.

Appendix A. Nonsmooth analysis and discontinuous vector fields.

In this appendix we review some basic facts and standard notations from nonsmooth analysis [4].

DEFINITION A.1. *A function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be* locally Lipschitz near *$x \in \mathbb{R}^N$ if there exist positive constants L_x and ϵ such that $|f(y) - f(y')| \leq L_x \|y - y'\|$ for all $y, y' \in B_N(x, \epsilon)$.*

Note that continuously differentiable functions at x are locally Lipschitz near x. The usual *right directional derivative* and the *generalized directional derivative* of f

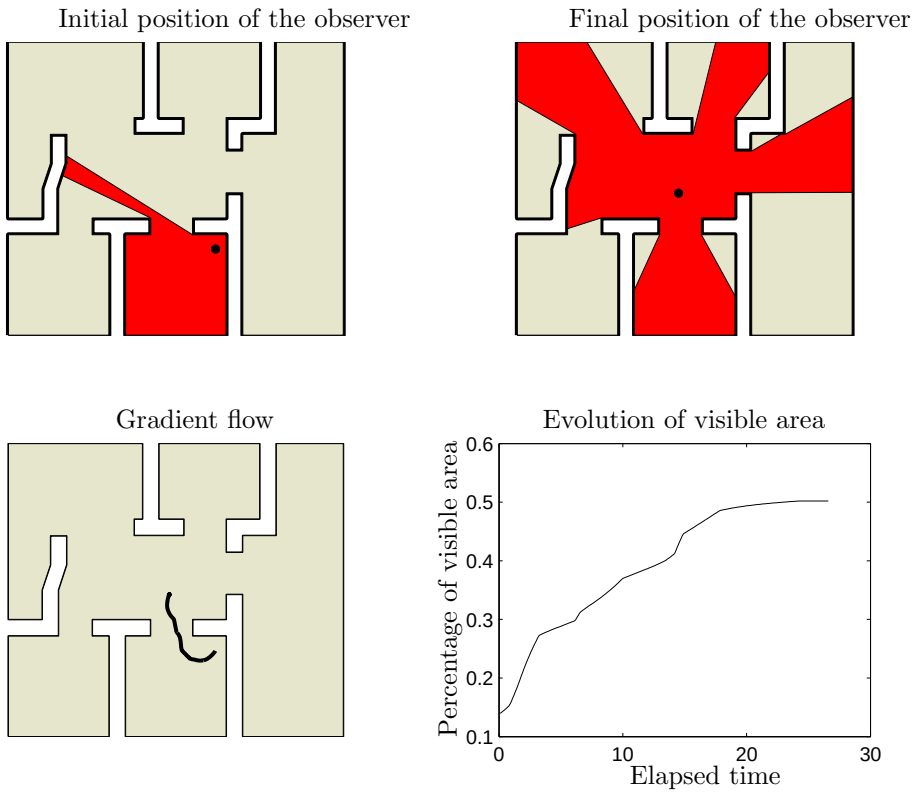

FIG. 5.4. *Simulation results of the gradient algorithm for the nonconvex polygon in Figure 1.1. The observer arrives, in finite time, at a local maximum.*

at x in the direction of $v \in \mathbb{R}^N$ are defined, respectively, as

$$f'(x;v) = \lim_{t \to 0^+} \frac{f(x+tv) - f(x)}{t}, \qquad f^o(x;v) = \limsup_{\substack{y \to x \\ t \to 0^+}} \frac{f(y+tv) - f(y)}{t}.$$

The limit in the definition of $f'(x;v)$ does not always exist, whereas the limit in $f^o(x;v)$ is always well-defined.

DEFINITION A.2. *A function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be* regular at $x \in \mathbb{R}^N$ *if for all $v \in \mathbb{R}^N$, $f'(x;v)$ exists and $f^o(x;v) = f'(x;v)$.*

Again, a continuously differentiable function at x is regular at x. Also, a locally Lipschitz function at x which is convex is also regular (cf. Proposition 2.3.6 in [4]).

From Rademacher's Theorem [4], we know that locally Lipschitz functions are continuously differentiable almost everywhere (in the sense of Lebesgue measure). If Ω_f denotes the set of points in $\mathbb{R}^N$ at which f fails to be differentiable, and S denotes any other set of measure zero, the *generalized gradient* of f is defined by

$$\partial f(x) = \operatorname{co}\left\{ \lim_{i \to +\infty} \mathrm{d}f(x_i) \mid x_i \to x\,,\ x_i \notin S \cup \Omega_f \right\}.$$

Note that this definition coincides with $\mathrm{d}f(x)$ if f is continuously differentiable at x. The generalized gradient and the generalized directional derivative (cf. Proposition 2.1.2 in [4]) are related by $f^o(x;v) = \max\{\zeta \cdot v \mid \zeta \in \partial f(x)\}$, for each $v \in \mathbb{R}^N$.

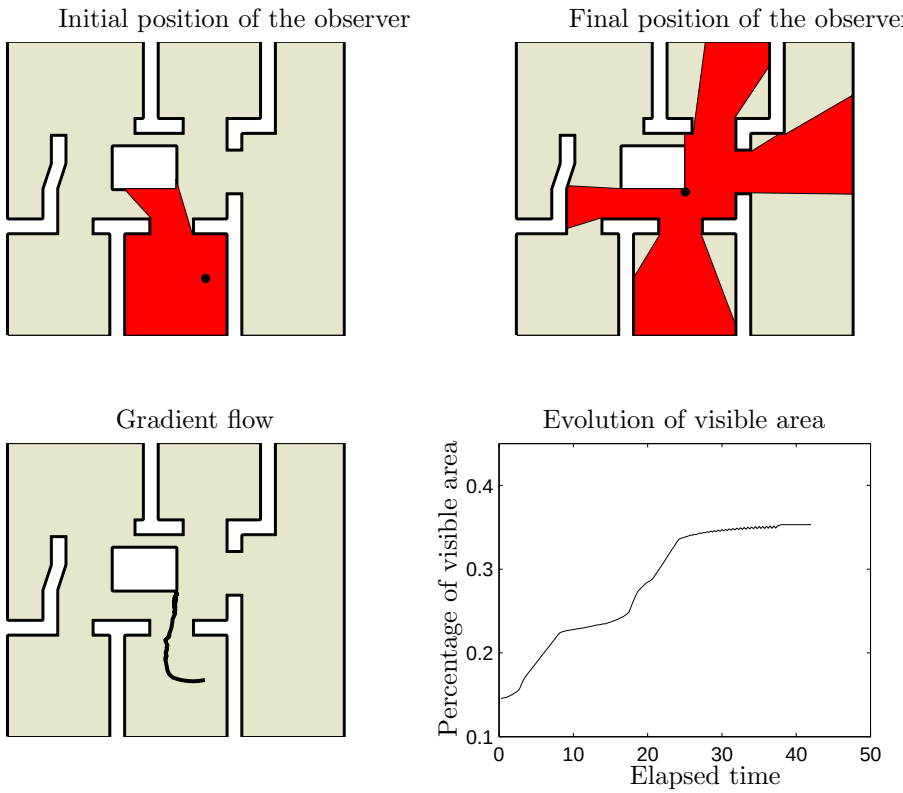

FIG. 5.5. *Simulation results of the gradient algorithm for an observer in a nonconvex environment with a hole. The observer arrives, in finite time, at a reflex vertex.*

A point $x \in \mathbb{R}^N$ which verifies that $0 \in \partial f(x)$ is called a *critical point of* f. The extrema of Lipschitz functions are characterized by the following result.

PROPOSITION A.3. *Let f be a locally Lipschitz function at $x \in \mathbb{R}^N$. If f attains a local minimum or maximum at x, then $0 \in \partial f(x)$, i.e., x is a critical point.*

Let $\mathrm{Ln} : 2^{\mathbb{R}^N} \to 2^{\mathbb{R}^N}$ be the set-valued map that associates to each subset S of $\mathbb{R}^N$ the set of its least-norm elements $\mathrm{Ln}(S)$. For a locally Lipschitz function f, we consider the *generalized gradient vector field* $\mathrm{Ln}(\partial f) : \mathbb{R}^N \to \mathbb{R}^N$ given by $x \mapsto \mathrm{Ln}(\partial f)(x) = \mathrm{Ln}(\partial f(x))$.

THEOREM A.4. *Let f be a locally Lipschitz function at x. Assume that $0 \notin \partial f(x)$. Then, there exists $T > 0$ such that $f(x - t\,\mathrm{Ln}(\partial f)(x)) \leq f(x) - \frac{t}{2}\|\mathrm{Ln}(\partial f)(x)\|^2$, $0 < t < T$. The vector $-\mathrm{Ln}(\partial f)(x)$ is called a* direction of descent.

For differential equations with discontinuous right-hand sides we understand the solutions in terms of differential inclusions following [8]. Let $F : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ be a set-valued map. Consider the differential inclusion

$$\dot{x} \in F(x)\,. \tag{A.1}$$

A solution to this equation on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as an absolutely continuous function $x : [t_0, t_1] \to \mathbb{R}^N$ such that $\dot{x}(t) \in F(x(t))$ for almost all $t \in [t_0, t_1]$. Given $x_0 \in \mathbb{R}^N$, the existence of at least a solution with initial condition x_0 is guaranteed by the following lemma.

LEMMA A.5. *Let the map F be upper semicontinuous with nonempty, compact and convex values. Then, given $x_0 \in \mathbb{R}^N$, there exists at least a solution of* (A.1) *with initial condition x_0.*

Now, consider the differential equation

$$\dot{x}(t) = X(x(t))\,, \tag{A.2}$$

where $X : \mathbb{R}^N \to \mathbb{R}^N$ is measurable and essentially locally bounded. We understand the solution of this equation in the Filippov sense. For each $x \in \mathbb{R}^N$, consider the set

$$K[X](x) = \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \mathrm{co}\{X(B_N(x,\delta) \setminus S)\}\,,$$

where μ denotes the usual Lebesgue measure in $\mathbb{R}^N$. Alternatively, one can show [13] that there exists a set S_X of measure zero such that

$$K[X](x) = \mathrm{co}\left\{ \lim_{i\to+\infty} X(x_i) \mid x_i \to x\,,\ x_i \notin S \cup S_X \right\}\,,$$

where S is any set of measure zero. A Filippov solution of (A.2) on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as a solution of the differential inclusion

$$\dot{x} \in K[X](x)\,. \tag{A.3}$$

Since the set-valued map $K[X] : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is upper semicontinuous with nonempty, compact, convex values and locally bounded (cf. [8]), the existence of Filippov solutions of (A.2) is guaranteed by Lemma A.5. A set M is *weakly invariant* (respectively *strongly invariant*) for (A.2) if for each $x_0 \in M$, M contains a maximal solution (respectively all maximal solutions) of (A.2).